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THE METHOD OF MONODROMIE WITH APPLICATIONS TO THREE PARAMETER QUARTIC EQUATIONS.

By R. P. Baker.

§ 1. The Technique of Monodromie.

The fundamental theorem on this subject is given by Jordan in his Traité des Substitutions, p. 277. It is there shown that, if $F(x, k) = 0$ is an algebraic equation, with coefficients in a domain including k, and has the algebraic group G, then a group of substitutions H can be found such that every rational function of the roots and k which is invariant under G is invariant under H; that H is an invariant subgroup under G and that an algebraic number can always be found whose adjunction reduces G to H. Moreover the method of determining H and this algebraic number are formally set out.

For the determination of H the parameter is allowed to vary in such a way that its path on the Neumann sphere for the complex variable k is a closed curve which does not pass through the branch points. Such a variation of k may cause the roots $(x_1, x_2, \cdots x_n)$ to be permuted according to a certain substitution which is an element of a group H generated by the totality of such substitutions. H is called the monodromie group.

The method of determining the algebraic number t whose adjunction reduces G to H is as follows: If φ is a function of the roots which is invariant under H and changed by every other substitution, it is a rational function of k, say $f(k)$. Replace all the coefficients of $f(k)$ by indeterminates and substitute the expression in the irreducible equation for φ, and then express the fact that the result is zero for every k. A certain number of algebraic equations are obtained with coefficients in R, which must be satisfied by the coefficients of $f(k)$. These coefficients are the roots of numerical equations in R, and their adjunction reduces G to H. This adjunction may be replaced by the adjunction of a single algebraic number, or by adjunction of all the roots of a single equation. From the latter method it follows that H is invariant under G.

The effective determination of H calls for a method of determining a finite number of elements, which generate the whole group. This is the object of this paper, the illustrations being taken from quartic equations. Incidentally the determination of t is illustrated.

§ 2. The Effective Determination of the Monodromie Group.

In the case of one parameter, k, the discriminant of the equation is a polynomial in k. At only a finite number of points are the roots equal. These points must be deleted from the domain of k, for otherwise the roots lose their identity and no question of a substitution can arise. Consider a path for k made up of a line from k_0, a non-discriminantal point, to a point arbitrarily near k_1, a point at which two roots are equal, a circle round k_1 and a return on the line to k_0. The line and the return can produce no substitution since the roots are all distinct and return to their original positions. The circle may or may not give a single transposition.

If in the neighborhood of k_1 the equation can be approximately expressed in the form

$$[(x - x_1)^2 - p(k - k_1)^n]G(x) = 0,$$

where n (> 1) is the index of the lowest power of $(k - k_1)$ entering, and $G(x)$ is finite as (x, k) approaches (x_1, k_1), then two cases arise:—

If n is an even integer $2m$, the first term reduces and, as $(k - k_1)$ expressed in the form ρ Exp $i(\varphi + \alpha)$ varies as φ increases from 0 to 2π, the two points representing the values of $(x - x_1)$, which are as nearly as we please $\pm \sqrt{\rho p} \cdot \text{Exp}\, mi(\varphi + \alpha)$, each make m circuits round the origin and are restored to their original position. In this case no substitution occurs.

If however n is an odd integer $2m + 1$, the values of $(x - x_1)$ are as nearly as we please $\pm \sqrt{\rho p} \cdot \text{Exp}\, (m + \frac{1}{2})i(\varphi + \alpha)$ and each makes m circuits and a half circuit, and the two roots suffer a simple interchange. The remaining roots, satisfying as nearly as we please the equation $G(x) = 0$, are finitely separated and remain each within a region that may be made as small as we please, and so cannot be interchanged.

In the general case a discriminantal value of k may cause several sets of roots to become equal, the irreducible binomials giving the cycles of the substitutions.

The most convenient method of determining the proper approximation is the Newton parallelogram. It may happen that a determination of the branch to a higher order is necessary, the details being worked out as in Chrystal's Algebra, vol. II, p. 383.

§ 3. A General Theorem concerning Monodromie.

Single transpositions occur at ordinary points on the discriminantal locus; all other substitutions including the identity occur at singular points.

If the roots are $x_1, x_2, x_3, \cdots x_n$ and Δ is the discriminant, the elementary symmetric functions being a_1, a_2, a_3, a_n, and if $\zeta(1, 2, \cdots n)$ denote the product of the differences of the roots taken in ascending order of subscripts, then giving a_1 an increment α and in consequence the roots increments $\xi_1, \xi_2, \cdots \xi_n$, we have, to the first order in small quantities,

$$\begin{aligned} \alpha &= \xi_1 + \xi_2 + \xi_3 + \cdots + \xi_n \\ 0 &= \xi_1 \cdot \Sigma x_2 + \xi_2 \cdot \Sigma x_1 + \cdots \\ 0 &= \xi_1 \cdot \Sigma x_2 x_3 + \xi_2 \cdot \Sigma x_1 x_3 + \cdots \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0 &= \xi_1 \cdot x_2 x_3 \cdots x_n + \xi_2 \cdot x_1 x_3 \cdots x_n + \cdots, \end{aligned}$$

where the subscripts of the x's omit the subscript of the ξ which multiplies their summations.

By solving this set of linear equations and proceeding to the limit, there results

$$\frac{\partial x_1}{\partial a_1} = \frac{x_1^{n-1}\zeta(2, 3, 4, \cdots n)}{\zeta(1, 2, 3, \cdots n)},$$

where the numerator ζ is defined as the same function of the set of roots omitting x_1 as the denominator is of the whole set.

In general by the same method we have

$$\frac{\partial x_j}{\partial a_k} = \frac{\zeta(1, 2, \cdots j-1, j+1, \cdots n)}{\zeta(1, 2, \cdots n)} \cdot x_j^{n-k}.$$

From this, since $\Delta = \zeta^2$,

$$\frac{\partial \Delta}{\partial a_k} = \Sigma \frac{\partial \Delta}{\partial x_j}\frac{\partial x_j}{\partial a_k} = 2\Sigma \frac{\partial \zeta}{\partial x_j} x_j^{n-k}\zeta(1, 2, \cdots j-1, j+1, \cdots n).$$

If more than two roots are equal or if there are two or more pairs of equal roots, every one of the partial derivatives in the last expression vanishes, and therefore every partial derivative of Δ with respect to the coefficients vanishes. Hence the point is singular on Δ. This establishes a necessary condition for substitutions other than single transpositions.

To arrive at sufficient conditions we suppose that the factor Δ_1 of the discriminant vanishes at the point $a_1', a_2', \cdots a_n'$ and the roots in its neighborhood form certain sets of equal roots. Since the values of the roots and the coefficients are analytic functions of Δ_1 with convergent series for small enough values of Δ_1, the equation can be expressed as nearly as we please in the form

$$F(x) = [(x - x_1)^{\alpha_1} - p_1\Delta_1^{\delta_1}][(x - x_2)^{\alpha_2} - p_2\Delta_1^{\delta_2}][\cdots] \cdots G(x) = 0,$$

where $G(x) = 0$ has no equal roots.

Taking first the case of the α's prime to the δ's, the vanishing terms of the discriminant in the field of the roots are contributed by differences of roots in the same cycle. If α is odd the discriminant of the binomial is even in Δ_1^δ and hence even in Δ_1. If α is even δ must be odd and the discriminant of the binomial is odd. Now the odd α's contribute odd cycles, equivalent to an even number of transpositions, and the even α's even cycles, hence if the whole substitution is odd the exponent of Δ_1 in the discriminant is odd; if even, even.

Taking next the case of an α and a δ having a common factor, we have several cycles with the same root value. Let these be

$$[(x - x_1)^{\alpha_1} - p_1\Delta_1^{\delta_1}][(x - x_1)^{\alpha_2} - p_2\Delta_1^{\delta_2}]\,[\cdots]\,\cdots.$$

The equalities among the roots of the same cycle obey the odd and even law as above and, if

$$\frac{\alpha_1}{\delta_1} \geqq \frac{\alpha_2}{\delta_2} \geqq \frac{\alpha_3}{\delta_3} \geqq \cdots,$$

the differences of the roots in the first cycle taken with those of the following cycles are in number $\alpha_1(\alpha_2 + \alpha_3 + \alpha_4 + \cdots)$ and of order δ_1/α_1, and the contribution from their squares to the order of Δ_1 in the discriminant is an even integer. The same holds of the pairs from the second cycle and those that follow it, and so on. The odd and even rule will be kept in this case and also if any number of root values have compound cycles. The rule then holds in general.

We notice that a pair of linear factors contribute at least Δ_1^2. This is the case corresponding to a nodal locus where no cycle exists.

Also if the δ's are known explicitly for all the cycles the exponent can be calculated; and that each set of cycles has a minimum exponent corresponding to the case where all the δ's are equal to unity. In the case of several parameters fixed values, not discriminantal and not causing reduction, may be given to all but one, and the cycles discovered by its variation are elements of the monodromie group. But if a discriminantal value or reducing value is assigned as fixed, in general cycles will be lost.

§ 4. **The General Quartic.**

By a rational transformation this can be brought to the form

$$x^4 + bx^2 - cx + d = 0. \tag{1}$$

The discriminant is

$$\Delta = 256d^3 + 16b^4d + 144bc^2d - 4b^3c^2 - 128b^2d^2 - 27c^4. \tag{2}$$

The discriminantal surface has the singular lines

$$c = 0, \qquad b^2 - 4d = 0,$$

and

$$b^2 + 12d = 0, \qquad 8b^3 + 27c^2 = 0.$$

The first is a locus of two pairs of equal roots, the second of three equal roots. The origin is a singular point at which four roots are equal. (Weber's Algebra, I, 249.) There are no other singularities.

To show that the corresponding cycles occur, we choose for the first line the point $c = 0$, $b = 2$, $d = 1$. The roots are $+i$, $-i$, each repeated. Writing $x = \xi + i$, $c = \gamma$, $b = 2 + \beta$, $d = 1 + \delta$, the equation becomes

$$\xi^4 + 4i\xi^3 + \xi^2(\beta - 4) + \xi(2i\beta - \gamma) + \delta - \beta = 0.$$

For $\beta = 0$, $\gamma = 0$, which are not in themselves discriminantal, the Newton parallelogram gives as proper approximation $-4\xi^2 + \delta = 0$ and a simple transposition of the pair of equal roots follows from the monodromie of δ round its origin. The same thing occurs for the other pair of equal roots.

It may be noticed that with $\beta = 0$, $\delta = 0$ reduction occurs and variations of γ can give no cycle.

For the locus of three equal roots we write $x = \xi + \sqrt{2}$, $b = -12$, $c = -16\sqrt{2}$ and $d = -12 + \delta$.

We have

$$\xi^4 + 4\sqrt{2}\xi^3 - \delta = 0,$$

and a 3-cycle. At the origin, writing $x = \xi$, $d = \delta$, $b = c = 0$, $\xi^4 + \delta = 0$ gives a 4-cycle.

Since no group but the symmetric group contains substitutions of these three types, the monodromie group is the symmetric group. In this case it follows at once that the algebraic group is the symmetric group.

§ 5. Quartics with the Alternating Group.

To find a general form for these, we express the fact that the discriminant is the square of a rational quantity and, by a chain of birational substitutions, reduce this equation to a form in which the coefficients can be determined as rational functions of prescribed form of three parameters.

The form thus arrived at will have generality of this kind, viz: It is a three parameter form, every special case of which has a square discriminant, and so a group G_{12}, or a subgroup, provided the discriminant is not zero. If as is usual irreducibility is postulated, it is general in the same sense as the general equation is in reference to its group. For every form with a non-vanishing square discriminant can be identified with this one. It turns out that the form possesses a property which the general equation does not, namely that a parameter can be removed by a rational transformation.

The discriminant can be written

$$16d(4d - b^2)^2 + 36bc^2(4d - b^2) + 32b^3c^2 - 27c^4 = t^2.$$

Putting $4d - b^2 = \rho$ and eliminating d, a birational operation, we have

$$t^2 = 4(b^2 + \rho)\rho^2 + (36b\rho + 32b^3 - 27c^2)c^2.$$

Write $t = 4nr$, $c = 2mr$, $\rho = 2r$ and eliminate c, t, ρ. The result, arranged as a quadratic in r, is then

$$27m^4r^2 - 2r(9bm^2 + 1) + n^2 - b^2 - 8m^2b^3 = 0.$$

If r is rational the discriminant must be a square.

Writing $3bm^2 = \eta$, $3nm^2 = \sigma$, this gives

$$8\eta^3 + 12\eta^2 + 6\eta + 1 - 3\sigma^2 = q^2,$$

or, if $2\eta + 1 = \theta$,

$$\theta^3 - 3\sigma^2 = q^2.$$

Apply now the birational transformation,

$$\sigma = s\theta, \qquad q = p\theta,$$

and obtain

$$\theta = 3s^2 + p^2.$$

Retracing the chain of transformations,

$$\begin{aligned} b &= \frac{(3s^2 + p^2 - 1)}{6m^2}, \\ c &= \frac{(3 + 2p)(3s^2 + p^2) - 1}{27m^3}, \\ d &= \frac{4[(3 + 2p)(3s^2 + p^2) - 1] + 3[3s^2 + p^2 - 1]^2}{432m^4}, \\ t &= \frac{2s(3s^2 + p^2)[(3 + 2p)(3s^2 + p^2) - 1]}{81m^6}. \end{aligned} \tag{3}$$

It is obvious that m can be removed by a rational transformation. In the course of the work the factors r^2 and θ^2 are rejected. Their vanishing leads to reduction.

To determine the monodromie group for this form, we remark that the parameter m is ineffective. Removing it by writing $x = \xi/6m$, and, in order to discuss the factor s of the discriminant, placing $p = 1$, we have in the neighborhood of the equal roots, if $\xi = 2 + y$ and $s = \sigma$, where y and σ are small,

$$81\sigma^4 + \sigma^2(18y^2 - 4\delta y + 12) + (y^4 + \delta y^3 + 24y^2) = 0$$

or, with the proper approximation

$$2y^2 + \sigma^2 = 0.$$

The locus $s = 0$ is then a nodal locus and gives no cycles. To test the locus $3s^2 + p^2 = 0$, we write $3s^2 = -1$, $p = 1 + \pi$, where π is small. The equal roots are at $\xi = 1$ and, writing $\xi = 1 + y$ where y is small, the proper approximation is

$$y^3 + 4\pi = 0.$$

Here three roots enter a cycle.

For the locus

$$(2p + 3)(3s^2 + p^2) - 1 = 0$$

take

$$p = 0, \qquad s = \tfrac{1}{3} + \lambda, \qquad \xi = \sqrt{2} + y.$$

The approximation is

$$y^2 + 3(3 - 2\sqrt{2})\lambda = 0.$$

This gives a cycle of two roots and similarly for the values $\xi = -\sqrt{2}$. The element is a double transposition. The only groups containing cycles of three and double transpositions are the symmetric group and the alternating group.

If in the investigation the exact form of the discriminant is known, the application of the method is trivial; if, however, as may be conceived, only the algebraic factors of the discriminant are known, the complete determination of the monodromie group requires the discussion of the intersection of the loci and their singularities. In this case a moderate amount of labor, which is left to the reader, shows that no further type of cycles arises and the fact that all the triple cycles and double transpositions occur in the alternating group obviates any need of identifying the roots.

With the given form the algebraic group is the alternating group. If we write $s = \sqrt{2}\sigma$ the coefficients will not contain the square root but the algebraic group is now the general group, reducing on adjunction of $\sqrt{2}$ to the alternating group in accordance with Jordan's theorem.

§ 6. Quartic Equations with the Group G_8.

The equation being as before (1), and the function $x_1x_2 + x_3x_4$ being rational, the equation satisfied by this and its conjugates, $x_1x_3+x_2x_4=t_2$, $x_1x_4 + x_3x_2 = t_3$, is

$$t^3 - bt^2 - 4dt + 4bd - c^2 = 0.$$

This is linear in b and the general equation with this group can be written

$$x^4 + \left(t - \frac{c^2}{\rho}\right) x^2 - cx + \left(\frac{t^2 - \rho}{4}\right) = 0, \tag{4}$$

since no restrictions on rationality are introduced by the transformation $t^2 - 4d = \rho$ which is birational in (d, ρ).

The discriminant is equal to the discriminant of the resolvent for

$$\begin{aligned} t_1 - t_2 &= (x_1 - x_4)(x_2 - x_3), \\ t_2 - t_3 &= (x_1 - x_2)(x_3 - x_4), \\ t_3 - t_1 &= (x_1 - x_3)(x_4 - x_2). \end{aligned}$$

But $(t_2 - t_3)^2$ can be expressed rationally, hence the discriminant reduces in the coefficients.

Corresponding to this factor we have Δ_1/ρ^2, where

$$\Delta_1 = 4t^2\rho^2 - 4c^2t\rho + c^4 - 4\rho^3. \tag{5}$$

The remaining factor is $\Delta_2{}^2 \div \rho^2$, where

$$\Delta_2 = \rho^2 + 2c^2t. \tag{6}$$

The complete discriminant is

$$D = \Delta_1 \cdot \Delta_2{}^2 \div \rho^4. \tag{7}$$

There are no singularities on either Δ_1 or Δ_2 at which $\rho \neq 0$. The discussion of monodromie may be limited to points on Δ_1, Δ_2, $\rho = 0$ and their intersections, and also $\rho = \infty$.

The locus $\Delta_2 = 0$ is a nodal locus and contributes no cycles or substitutions. The approximation at the point ($\rho = 2$, $t = -2$, $c = 1 + \gamma$, $x = \xi - 1$) is

$$7\xi^2 + 2\gamma\xi - \gamma^2 = 0.$$

This character is kept over the whole locus except possibly at the intersections with the other discriminantal loci.

The locus $\Delta_1 = 0$ produces a single transposition; the point ($\rho = 1$, $t = 3/2$, $c = 1 + \gamma$, $x = \xi + 1/2$ has for proper approximation

$$5\xi^2 - 2\gamma = 0.$$

The locus $\rho = 0$ gives a pair of transpositions, one pair of roots being zero and the other infinite. An exception arises if $c = 0$, for then $\Delta_2 = 0$ and no cycle is formed for a general approach.

The locus, $\rho = \infty$, gives a cycle of four roots with infinite values unless $c = 0$ when in general no cycle exists.

We may name the transposition on Δ_1 (12), and, by the continuity of the roots as we move along Δ_1 to $\rho = \infty$, $c \neq 0$, this pair of roots must take for a real path the real (opposite) positions in the cycle of four. This cycle is then (1324) or (1423).

With these elements the group G_8 is generated and it remains to be shown that no further elements can occur to raise this to G_{24}.

As the problem stands, the labor of discussing all the intersections, including the infinite values, is prohibitive but can be avoided.

By means of a birational transformation of the parameters and a rational transformation of the variable, a parameter can be eliminated. We write

$$2t = \epsilon\pi(1 + \pi\lambda^2), \qquad c = \epsilon\lambda\pi^2, \qquad \rho = \epsilon\pi^2,$$

whose inverses are

$$\lambda = \frac{c}{\rho}, \qquad \pi = \frac{\rho^2}{2}(t\rho - c^2), \qquad \epsilon = \frac{4(t\rho - c^2)^2}{\rho^3}.$$

This gives for the coefficients

$$2b = \epsilon\pi(1 - \pi\lambda^2), \qquad 16d = \epsilon\pi^2[\epsilon(1 + \pi\lambda^2)^2 - 4].$$

If we now write $x = \xi/\lambda$ and $\pi\lambda^2 = \alpha$ the equation becomes

$$16\xi^4 + 8\epsilon\alpha(1 - \alpha)\xi^2 - 16\epsilon\alpha^2\xi + \epsilon\alpha^2[\epsilon(1 + \alpha)^2 - 4] = 0. \tag{8}$$

and the discriminant is

$$D = \epsilon^3\alpha^6(\epsilon\alpha^2 + \epsilon\alpha + 1)^2(\epsilon - 4).$$

It is now easy to show that, for $\alpha = -1$, G_8 is generated by the substitutions at ϵ cycles. The monodromie group is then either G_8 or G_{24}. For $\epsilon = 1$, the monodromie group is G_2', the only cycle being a double transposition at $\alpha = 0$. By Jordan's theorem the algebraic group can now be only the cyclic group in the G_8 or reduction has occurred. As a matter of fact the equation reduces and the question is at once raised as to the adjunction of a square root. Writing $\epsilon = \varphi^2$, the equation reduces to $\left[\xi^2 + \varphi\alpha\xi + \frac{\varphi^2\alpha(1 + \alpha + 2\varphi)}{4}\right]$ and the conjugate factor.

The adjunction of a square root cannot reduce the group from G_{24}, except to G_{12} which is barred since odd substitutions occur. Hence the group is G_8 and the monodromie group is the same.

Although more facts were at hand, the result could be obtained from a knowledge of the algebraic factors of the discriminant apart from their exponents and a monodromie in which identification of the roots was not needed.

§ 7. Quartic Equations with the Group G_4: (1), (12) (34), (13) (24), (14) (23).

As functions belonging to the group we may take any one of

$$f_1 = (x_1 - x_2)(x_3 - x_4),\quad f_2 = (x_1 - x_3)(x_4 - x_2),\quad f_3 = (x_1 - x_4)(x_2 - x_3),$$

or their negatives.

If

$$t_1 = x_1x_2 + x_3x_4,\quad t_2 = x_1x_3 + x_2x_4,\quad t_3 = x_1x_4 + x_2x_3,$$

the t's are functions in G_8.

Since $f_1 = t_2 - t_3$ and the t's are roots of a cubic resolvent, then, if f_1 is rational, all the roots of this cubic are rational; for $t_2 - t_3$ is a six-valued function. It follows that all the f's are rational.

If the resolvent for t has one root, t_3, rational and $t_1 + t_2 = R$, the residual quadratic for t_1, t_2 is

$$T^2 - RT - \left(\frac{c^2}{R} + t_3^2 + Rt_3\right) = 0.$$

That this may have rational roots, the discriminant must be a square. This condition requires

$$R = 4c^2 \div (\kappa^2 - \lambda^2) \qquad \text{and} \qquad t_3 = \frac{(\lambda - R)}{2},$$

the discriminant being κ^2.

We may now take

$$t_1 = \frac{2c^2}{\kappa^2 - \lambda^2} + \frac{\kappa}{2},\qquad t_2 = t_1 - \kappa,\qquad t_3 = \frac{\lambda}{2} - \frac{2c^2}{\kappa^2 - \lambda^2},$$

$$f_1 = \frac{4c^2}{\kappa^2 - \lambda^2} - \frac{\kappa + \lambda}{2},\qquad f_2 = \frac{4c^2}{\kappa^2 - \lambda^2} + \frac{\kappa - \lambda}{2},\qquad f_3 = \kappa.$$

The square root of the discriminant of the original equation is $-f_1f_2f_3$, hence the discriminant is

$$\Delta \equiv \frac{\kappa^2[64c^4 - 16\lambda(\kappa^2 - \lambda^2)c^2 + (\lambda^2 - \kappa^2)^3]^2}{16(\kappa^2 - \lambda^2)^4}. \tag{9}$$

By means of the given relations the equation becomes

$$x^4 + x^2\left[\frac{4c^2 + \lambda(\kappa^2 - \lambda^2)}{2(\kappa^2 - \lambda^2)}\right] - cx + \frac{[\kappa^2(\kappa^2 - \lambda^2)^2 - 8\lambda c^2(\kappa^2 - \lambda^2) + 16c^4]}{16(\kappa^2 - \lambda^2)^2} = 0. \tag{10}$$

The roots are

$$x_1 = \frac{c}{\sqrt{(\lambda^2 - \kappa^2)}} + \tfrac{1}{2}\sqrt{[-\lambda + \sqrt{(\lambda^2 - \kappa^2)}]}$$

and the conjugates.

The discriminantal locus, $\kappa = 0$, has a pair of roots equal to c/λ, and the approximation is of the form

$$x^4 + x^3 + x^2 + \kappa^2 x + \kappa^2 = 0,$$

where the coefficients, in general not zero, have been suppressed. No cycle is obtained.

If $\kappa = \lambda$, the discriminant is infinite. To examine this locus, write $\kappa - \lambda = \alpha$, and the equation is of the form

$$\alpha^2 x^4 + \alpha x^2 - \alpha^2 x + c^4 = 0.$$

If we write $x = 1/\xi$, the terms to be retained are

$$\alpha^2 + \alpha\xi^2 + \xi^4,$$

giving two separate cycles among the infinite vaues of x.

For the special values $c = 0$, $\lambda = -4$, the equation takes the form

$$x^4 - 2x^2 + \frac{\kappa^2}{16} = 0,$$

a real curve of the general type of a lemniscate, and it is easy to see that two different double transpositions occur and that the monodromie group contains G_4. The cycles occur here when $\kappa = \pm 4$, that is, $\kappa = \pm \lambda$. To show that the remaining factors of the discriminant contribute nothing further, we remark that

$$64c^4 - 16\lambda c^2(\kappa^2 - \lambda^2) + (\lambda^2 - \kappa^2)^3$$

reduces to

$$[8c^2 - (\kappa^2 - \lambda^2)(\lambda - \kappa)]\,[8c^2 - (\kappa^2 - \lambda^2)(\lambda + \kappa)],$$

that is, $4f_2 f_3(\kappa^2 - \lambda^2)^2$. Since the f's are all rational, there is no real distinction between the properties of $f_1 = 0$ and of $f_2 = 0$. Hence these factors are nodal also and contribute no cycles. From the point of view of monodromie only, this can be established by a special non-singular point which may be conveniently taken as

$$\lambda = 0, \qquad \kappa = -2, \qquad c = 1 + \gamma, \qquad x = \tfrac{1}{2} + \xi.$$

The approximate form is $x^2 + 2x^3 + 2x^2 + \gamma^2/2\cdot x + \gamma^2/2$.

The detailed examination of the singularities of the discriminant may be shortened by the consideration that the intersection of two nodal loci cannot furnish cycles. This leaves the intersection of the effective loci $\kappa - \lambda = 0$, $\kappa + \lambda = 0$ as the only case to be examined for intersections and this with $c = 0$ as the only singularity on the nodal loci. It is easily shown that in no case does any other kind of substitution arise except double transposition.

If either of the quantities $\kappa - \lambda$, $\kappa + \lambda$ are restricted by being placed equal to a fixed rational quantity, the only remaining substitution is a double transposition, and the monodromie group reduces to G_2, namely 1, (12)(34). If both are so restricted, that is, if κ, λ are constants, the variation of c produces no cycles and the monodromie group is the identity.

A two-parameter form for this case is obtainable from that for G_8 (8) by expressing the discriminant as a square.

The only non-square factor is $\epsilon(\epsilon - 4)$ and, by writing $\epsilon = 4/(1 - \lambda^2)$, we have the form

$$(1-\lambda^2)^2\xi^4 + 2\alpha(1-\alpha)(1-\lambda^2)\xi^2 - 4\alpha^2(1-\lambda^2)\xi + \alpha^2(\alpha^2+\lambda^2+2\alpha) = 0 \quad (11)$$

with the discriminant

$$4\lambda^2(1 - \lambda^2)^6\alpha^6[(2\alpha + 1)^2 - \lambda^2]^2.$$

The monodromie is managed by following the plan for the two-parameter form of the G_8 equation noting that the transformation from ϵ to λ destroys the single transpositions.

§ 8. The Reciprocal Quartic.

The equation being

$$x^4 - ax^3 + bx^2 - ax + 1 = 0, \quad (12)$$

the discriminant is

$$\Delta = - [a^2 - 4(b - 2)]^2[4a^2 - (b + 2)^2]. \quad (13)$$

If we assume irreducibility, the group may be determined thus: For a proper choice of root names, we have

$$x_1x_2 + x_3x_4 = 2.$$

The case of two conjugates being equal to 2 leads to the vanishing of the first factor of the discriminant, and, if one is 2 and the others are equal, the second factor vanishes. The group is then G_8 or a subgroup. The monodromie group contains a single transposition at $a = 1$, $b = 0$. The monodromie group and the algebraic group are then both G_8.

To show the inclusion of this case in the general form for G_8, we write the equation as

$$x^4 - 4ax^3 + 6bx^2 - 4ax + 1 = 0,$$

and remove the second term.

The result is

$$x^4 + 6(b - a^2)x^2 - 4a(2a^2 - 3b + 1)x + (1 + 6a^2b - 4a^2 - 3a^4) = 0,$$

which is included in the general form if we write

$$t = 2(1 - a^2), \quad \rho = -8a^2(3b - 2a^2 - 1), \quad c = - 4a(3b - 2a^2 - 1).$$

To make a complete discussion by monodromie only, after establishing the single transposition, we consider a section of the locus by $b = 0$ and bring up for inspection complex points by writing

$$x = i\xi, \qquad a = -i\alpha.$$

The locus is then $\alpha = (1 + \xi^4)/\xi(1 - \xi^2)$ and the graph makes visible the two distinct double transpositions which are needed to generate the group G_8. The real graph shows that the two single transpositions may be given the same notation.

§ 9. Quartic Equations Solved by Quadratic Forms.

By a linear transformation these may be taken as

$$(x^2 + \beta)^2 + 2m(x^2 + \beta) + n = 0. \tag{14}$$

The discriminant is $64(m^2 - n)^2[(\beta + m)^2 - (m^2 - n)]$. If $\beta + m = -p$ and $m^2 - n = q$, the roots are

$$x_1 = \sqrt{p + \sqrt{q}}, \qquad x_2 = \sqrt{p - \sqrt{q}}, \qquad x_3 = -x_1, \qquad x_4 = -x_2.$$

The function $x_1x_3 + x_2x_4 = -2p$ is, in G_8, rational and distinct from its conjugates. The group is G_8 or a subgroup. Assuming irreducibility, it is G_8 if single transpositions occur.

For $\beta = 1$, $m = 1$, $n = 1 + \lambda$ and $x = \xi + i$, the approximation is

$$\xi^4 + 4i\xi^3 - 4\xi^2 + \lambda = 0$$

giving a 2-cycle and two distinct roots. The group is therefore G_8. The general form for G_8 equations includes this if

$$t = 2(\beta + m), \qquad c = 0, \qquad \rho = 4(m^2 - n).$$

To show by monodromie only that the group is at least G_8, we take the special case $\beta = 0$, when the equation reads

$$x^4 + 2mx^2 + n = 0.$$

If $m = 0$, we have a 4-cycle at the origin, if $m \neq 0$, a 2-cycle. By the continuity of the roots we see that these may be denoted by (1324) and (12) which generate G_8. To show that the group is not higher, we use the method of reduction of parameters. The special form just discussed is general by a birational transformation and change of notation.

§ 10. Quartic Equations with the Cyclic Group.

As a function belonging to $C_4[1 : (1234) : (13)(24) : (1432)]$, we take

$$(x_1 - x_3)(x_2 - x_4)[(x_1 - x_3)^2 - (x_2 - x_4)^2] = 4\lambda.$$

Assuming $a = x_1 + x_2 + x_3 + x_4 = 0$ and writing $x_1x_3 + x_2x_4 = t$, a function in G_8, we have

$$\lambda^2 = [t^2 - 4d][(t - b)^2 - 4t(t - b) + 16d], \tag{15}$$

where t is a rational solution of

$$t^3 - bt^2 - 4dt + 4bd - c^2 = 0. \tag{16}$$

The resolvent equation for λ, found by eliminating t, is

$$\begin{aligned}\lambda^6 - \lambda^4(16b^2d - 6bc^2 - 64d^2) + \lambda^2(9b^2c^4 - 384bc^2d - 32b^3c^2d + 108c^4d)\\ - c^4[256d^3 + 16b^4d + 144bc^2d - 4b^3c^2 - 128b^2d^2 - 27c^4] = 0.\end{aligned} \tag{17}$$

The last bracket contains the discriminant of the original equation.

To obtain the general equations of this group we write $t^2 - 4d = \rho$ and replace d in (15) and (16). This is a birational transformation as far as d and ρ are concerned.

From (16) we have $b = t - c^2/\rho$, and substitution in (15) gives

$$\lambda^2 = \frac{(2t\rho - c^2)^2}{\rho} - 4\rho^2.$$

Replace t by κ by the birational transformation $2t\rho - c^2 = \kappa\rho$, and we have $\lambda^2 + 4\rho^2 = \kappa^2\rho$. Using $\lambda = \sigma\rho$ to replace λ by σ, we have $\sigma^2\rho = \kappa^2 - 4\rho$. The factor ρ is removed in the operations, as $\rho = 0$ involves reduction. Solving for ρ and retracing the chain, we obtain b, d in terms of κ, σ, c, and the equation is

$$x^4 + x^2\left[\frac{\kappa^3 - c^2(\sigma^2 + 4)}{2\kappa^2}\right] - cx + \frac{[\kappa^3 + c^2(\sigma^2 + 4)]^2}{16\kappa^4} - \frac{\kappa^2}{4(\sigma^2 + 4)} = 0. \tag{18}$$

The discriminant for the integral form is

$$\Delta = \kappa^{18} \cdot (\sigma^2 + 4)^3 \cdot \sigma^2[\kappa^6 + (\sigma^2 + 4)^2\kappa^3c^2 + (\sigma^2 + 4)^3c^4]^2,$$

and is wholly singular as it should be.

It is obvious that, if $\sigma^2 + 4$ is a square, the group reduces to G_2. In this case the equation reduces to two quadratics

$$x^2 + \frac{ctx}{\kappa} + \frac{t(\kappa^2 + c^2t^2) - 2\kappa^3}{4\kappa^2t} = 0$$

and the conjugate, where

$$t^2 = \sigma^2 + 4.$$

As a check we notice that the cyclotomic equation is included for the values

$b = 5/8$, $c = -5/8$, $d = 205/256$, $t = 15/8$, $\kappa = 5/2$, $\sigma = 4$, $\lambda = 5/4$. The group is C_4 for the conjugates are distinct, namely, if $\lambda = 0$ or if two values of t are equal, reduction occurs.

In the monodromie inquiry the nodal character of the squared bracket in the discriminant is conveniently established with the values $c = 10$, $\kappa = 1 + \epsilon$, $\sigma^2 = -81/20$ and $x = 1 + \xi$. For $\sigma = 2i$, four roots are infinite and a 4-cycle is formed; so also for the conjugate value of σ. Double transpositions occur at $\kappa = 0, \infty$ and at $\sigma = 0, \infty$.

The 4-cycle generates C_4 and the question whether the monodromie group is not higher remains.

As before the number of parameters may be reduced.

We write

$$c = \lambda\pi^2(\sigma^2 + 4), \qquad \kappa = \pi(\sigma^2 + 4),$$

and replace c, κ by π, λ which is a birational operation. Writing $x = \xi\sqrt{\lambda}$ and $\pi\lambda^2 = \alpha$ we have the form

$$16\xi^4 + 8\alpha(1 - \alpha)(\sigma + 4)\xi^2 - 16\alpha^2(\sigma^2 + 4)\xi \\ + \alpha^2(\sigma^2 + 4)[(1 + \alpha)^2(\sigma^2 + 4) - 4] = 0 \tag{19}$$

with the discriminant

$$\Delta = (\sigma^2 + 4)^3\sigma^2\alpha^6[(\sigma^2 + 4)(\alpha^2 + \alpha) + 1]^2.$$

This differs from the two-parameter form for G_8 only by ϵ being replaced by $\sigma^2 + 4$. The single transposition of the G_8 form is now lost and the argument can be made as in the case of G_8.

We may notice as a convenient form of the resolvent equation for C_4 the condition that

$$16dt^2 - 3c^2t - 64d^2 - bc^2 \equiv \lambda^2$$

is a perfect square. This is derived by a partial elimination from (15) and (16).

§ 11.

As an explicit example of Jordan's method for determining the algebraic number whose adjunction reduces the group of the equation to the same group as the monodromie group, the case of a monodromie group G_4 under G_{12} may be taken.

To construct such an equation we notice that the factor $3s^2 + p^2$ of the discriminant of the G_{12} form (3) gives rise to 3-cycles. To obtain a G_4 monodromie group this must be obviated. It is sufficient for example to write $p = s$ The resulting form is

$$x^4 + 6(4s^2 - 1)x^2 - 8(8s^3 + 12s^2 - 1)x + 3(48s^4 + 32s^3 + 24s^2 - 1) = 0 \tag{20}$$

and the monodromie group is G_4.

The resolvent equation for G_4 when the discriminant Δ is a square is

$$f^3 - (12d + b^2)f - \sqrt{\Delta} = 0$$

in this case

$$b^2 + 12d = 2^6 \cdot 3^2(4s^2 + 2s + 1)s^2,$$

$$\sqrt{\Delta} = 2^9 \cdot 3^2(8s^3 + 12s^2 - 1)s^3.$$

By Jordan's theorem an algebraic field exists in which f is rational and in this special case all three f's are rational if one is (§ 7).

Writing $f = 8\varphi s$, we have for φ

$$\varphi^3 - 9(4s^2 + 2s + 1)\varphi - 9(8s^3 + 12s^2 - 1) = 0.$$

If now $\varphi = as + b$ and the equation is identically satisfied in s, we have

$$a^3 - 36a - 72 = 0, \qquad a^2b - 12b - 6a - 3b = 0,$$

$$ab^2 - 6b - 3a = 0, \qquad b^3 - 9b + 9 = 0.$$

These equations must by the general argument have consistent solutions.

Solving the equation for b we find

$$b_1 = 2\sqrt{3} \cos 50° = 2(\cos 80° + \cos 20°).$$

If ϵ is the primitive ninth root of unity, $\cos 40° + i \sin 40°$, and if $p_1 = \epsilon + \epsilon^8$, $p_2 = \epsilon^2 + \epsilon^7$, $p_3 = \epsilon^4 + \epsilon^5$ are periods, then

$$b_1 = p_2 - p_3, \quad b_2 = p_3 - p_1, \quad b_3 = p_1 - p_2.$$

The equation for a gives

$$a_1 = 2(p_3 - p_2), \quad a_2 = 2(p_1 - p_3), \quad a_3 = 2(p_2 - p_1).$$

By forming a table of multiplication for the periods

$$p_1^2 = 2 + p_2, \qquad p_2^2 = 2 + p_3, \qquad p_3^2 = 2 + p_1,$$

$$p_2p_3 = -1 + p_2, \qquad p_3p_1 = -1 + p_3, \qquad p_1p_2 = -1 + p_1,$$

we find that the pairs (a_1, b_3), (a_2, b_1), (a_3, b_2) satisfy the second and third equation. In the field of ϵ the equation for φ, and therefore that for f, is completely deducible. This field is however not the minimum field. The field determined by any root of the equation for b is the minimum; for all the b's are rationally expressible in terms of any one of them. The a's are mere multiples of the b's. The b equation has the group G_3 and is its own Galois resolvent.

Further examples of the monodromie group being a subgroup can be made by the device of setting the discriminantal factor which gives cycles not in the subgroup equal to a fixed rational, or working on the subgroup form and keeping algebraic rationality while destroying the arithmetic rationality.

For G_4 under G_8, for instance, using the three parameter form and writing

$$\rho = p^2 - \alpha, \qquad t = p + \frac{c^2}{2(p^2 - \alpha)},$$

the factor Δ_1 of the discriminant which gives rise to the single transposition becomes $4\alpha\rho^2$. If now α is a fixed non square, the group does not reduce but the monodromie group does and to G_4 invariant under G_8.

Here

$$b = p - \frac{c^2}{2(p^2 - \alpha)}, \qquad d = \frac{c^4 + 4pc^2(p^2 - \alpha) + 4\alpha(p^2 - \alpha)^2}{16(p^2 - \alpha)^2}.$$

For $c = 0$, $\alpha = 2$, the equation is

$$x^4 + px^2 + \tfrac{1}{2} = 0,$$

and the monodromie is obvious from the real graph. The adjunction of $\sqrt{\alpha}$ reduces the group to the monodromie group.

For the case of C_4 under G_8, we notice that the C_4 function λ is connected with the G_8 form in this way,

$$\lambda^2 = \frac{\Delta_1}{\rho}.$$

If Δ_1/ρ is a square, the group is C_4. If Δ_1 is a square the group is G_4, while if both these forms are squares the group is G_2 and reduction occurs.

If in the form for G_8 we write

$$\Delta_1 = k\rho^3\varphi^2, \qquad c = \gamma\rho, \qquad t = h\rho,$$

there results for the new form

$$b = \frac{(h + \gamma^2)(4 + k\varphi^2)}{(\gamma^2 - 2h)^2}, \qquad d = \frac{[h^2(4 + k\varphi^2) - (\gamma^2 - 2h)^2](4 + k\varphi^2)}{4(\gamma^2 - 2h)^4}.$$

If k is a non square fixed rational, the monodromie group is C_4 and the group G_8. Adjunction of $\sqrt{k}$ reduces the group to C_4.

For an example of H_4 under G_8, using the two-parameter form for G_8, we write

$$\epsilon = -\frac{2(\kappa^2 - 2\lambda - 1)}{\lambda^2}, \qquad \alpha = -\frac{2}{\lambda^2 - 2\lambda - 1}$$

and $x = \xi/\lambda$. The equation becomes

$$\xi^4 + 2(\lambda - 1)^2\xi^2 + 8\lambda\xi + (\lambda^4 - 4\lambda^3 + 6\lambda^2 + 4\lambda + 1) = 0.$$

This is derived from two circles

$$(x - a)^2 + y^2 - 2ay = 0,$$

$$(x - b)^2 + y^2 - 2by = 0,$$

where

$$a = 1 + i, \qquad b = 1 - i, \qquad y = \lambda.$$

The monodromie is evident from the geometry of the real case. The adjunction of i reduces the group to H_4.